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8 p.

Calculations of Real-Gas Effects in Flow Through Critical-Flow Nozzles

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Computer calculations have been made of how real-gas effects modify the conventional one-dimensional equations for mass flow of air, nitrogen, oxygen, hydrogen, argon, helium, and steam through a nozzle. The results indicate that for critical flow of air, at room temperature and 100 atmospheres pressure, real-gas effects of $3^1/_2$ percent exist. Similar magnitudes are found for the other gases.

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Introduction

One of the difficulties in the conventional use of critical-flow nozzles for metering the mass-flow of gases is the lack of accurate one-dimensional flow relations that take into account such "real-gas" effects as compressibility and specific-heat variations. Most published critical-flow equations apply to a perfect gas. For this report, a perfect gas will be defined as one whose specific heat is constant and independent of temperature and pressure, and whose compressibility factor Z is constant. At high pressures or low temperatures (for example, values of pressure and temperature near the critical point) significant errors in mass-flow-rate calculation can occur if the ideal-gas flow relations are used. Ideal-gas flow relations are defined as those that apply to a perfect gas.

References [1 to 3][1] estimate the values of gaseous imperfections for isentropic flow by use of the van der Waals and the Beattie-Bridgeman state equations. Reference [4] is an experimental check on the results of references [2 and 3]. An "effective gamma" for the isentropic expansion of real gases is estimated in reference [5]. Reference [6] presents a graphical method for computing the mass-flow rate for real gases by means of the data from reference [7].

In this paper, the mass-flow rates at the critical pressure ratio of air, nitrogen, and oxygen are calculated with the compressibility and ideal-gas specific-heat data from reference [7]. These mass-flow rates are also calculated for normal hydrogen by means of the data from reference [8] and for steam by means of the state equation in reference [9]. References [8 and 9] are the sources for the compressibility data for normal hydrogen and steam in reference [7]. In addition, less exact calculations are performed for all the above gases except steam with the Beattie-Bridgeman equation of state and the specific-heat data of reference [7]. The Beattie-Bridgeman constants are from reference [10]. This permits a check on the accuracy achievable by using the Beattie-Bridgeman equation. In addition, the mass-flow rates of argon and of helium are computed with the Beattie-Bridgeman equation.

For all gases except steam, the calculations are for temperatures up to 700 deg R and for pressures up to 100 atm. For steam, the ranges are 1500 deg R and 300 atm.

Calculation Procedure

Equations for calculating the mass-flow rate will be derived for two situations. One set will be derived for the situation where the compressibility factor Z is given as a function of pressure and temperature. The other set will be derived for the situation where Z is given as a function of density and temperature. The only other variable needed for these calculations is the ideal-gas specific heat c_{p0}, which is a function of temperature only. Both of these methods are required due to the manner in which the data of reference [7] are presented for different gases. The two cases will now be discussed.

Case I: $Z = Z(p, T)$.

This case applies to nitrogen and to oxygen. In reference [7], the compressibility factor is expressed as

$$Z(p, T) = \frac{p}{\rho RT} = 1 + B_p p + C_p p^2 + D_p p^3 \qquad (1)$$

where B_p, C_p, and D_p are tabulated as functions of temperature. For the purposes of these calculations, analytic expressions had to be found to represent B_p, C_p, and D_p. This was done by fitting temperature polynomials to the tabulated data. To cover the required temperature range, different equations were needed over different segments of the range. At places where two segments met, it was required that the function and its first and second derivatives be continuous. A temperature polynomial was also

[1] Numbers in brackets designate References at end of paper.

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Nomenclature

- A = Beattie-Bridgeman const
- a = Beattie-Bridgeman const
- B = Beattie-Bridgeman const
- B_p = second virial coefficient for pressure series
- B_ρ = second virial coefficient for density series
- b = Beattie-Bridgeman const
- C = Beattie-Bridgeman const
- C_p = third virial coefficient for pressure series
- C_ρ = third virial coefficient for density series
- c_p = specific heat at constant pressure
- c_{p0} = ideal-gas specific heat at constant pressure
- c_v = specific heat at constant volume
- D_p = fourth virial coefficient for pressure series
- D_ρ = fourth virial coefficient for density series
- H = enthalpy
- M_1 = Mach number
- p = pressure
- R = specific gas constant
- S = entropy
- T = temperature
- U = internal energy
- V = velocity
- Z = compressibility factor, $p/\rho RT$
- α = sound velocity
- γ = specific-heat ratio
- γ_i = 4/3 for steam; 7/5 for diatomic gases; 5/3 for monatomic gases
- ρ = density

Subscripts

- 0 = inlet stagnation condition
- 1 = nozzle throat conditions

used to represent the normalized ideal-gas specific heat c_{p0}/R, as tabulated in reference [7].

The given values for the calculation are the inlet stagnation pressure p_0, the inlet stagnation temperature T_0, and the Mach number at the nozzle throat M_1. By assuming the gas to be perfect, an initial estimate of the nozzle-throat temperature T_1 can be made. The actual Mach number for this estimate is computed as follows.

The differential entropy and enthalpy (reference [11]) are

$$dS = c_p \frac{dT}{T} + \frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_p dp \quad (2)$$

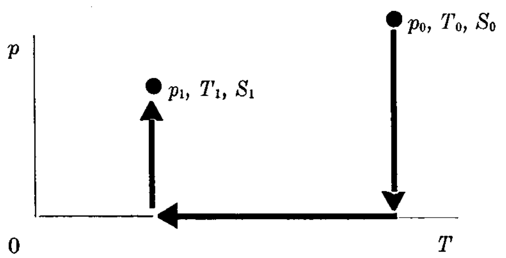
$$dH = T\,dS + \frac{1}{\rho}dp = -V dV \quad (3)$$

If the density ρ is eliminated from these equations by means of equation (1), the result is

$$\frac{dS}{R} = \frac{c_p}{R}\frac{dT}{T} - \left[Z + T\left(\frac{\partial Z}{\partial T}\right)_p\right]\frac{dp}{p} \quad (4)$$

$$\frac{dH}{R} = \frac{c_p}{R}dT - T^2\left(\frac{\partial Z}{\partial T}\right)_p \frac{dp}{p} \quad (5)$$

Equations (4) and (5) were integrated along the path shown in the following sketch:



Equations (4) and (5) in integral form become

$$\frac{S_1 - S_0}{R} \equiv 0 = \ln\frac{p_0}{p_1} - \int_{p_0}^{0}\left[Z - 1 + T\left(\frac{\partial Z}{\partial T}\right)_p\right]_{T=T_0}\frac{dp}{p}$$
$$+ \int_{T_0}^{T_1}\frac{c_{p0}}{R}\frac{dT}{T} - \int_0^{p_1}\left[Z - 1 + T\left(\frac{\partial Z}{\partial T}\right)_p\right]_{T=T_1}\frac{dp}{p} \quad (6)$$

where c_{p0} is the value of the constant-pressure specific heat at zero pressure (also the ideal-gas specific heat), and

$$\frac{H_1 - H_0}{R} = -\frac{1}{2}\frac{V_1^2}{R} = -\int_{p_0}^{0}\left[T^2\left(\frac{\partial Z}{\partial T}\right)_p\right]_{T=T_0}\frac{dp}{p}$$
$$+ \int_{T_0}^{T_1}\frac{c_{p0}}{R}dT - \int_0^{p_1}\left[T^2\left(\frac{\partial Z}{\partial T}\right)_p\right]_{T=T_1}\frac{dp}{p} \quad (7)$$

With the aid of a high-speed digital computer, equations (6) and (7) were solved for p_1 and V_1. The computational accuracy of p_1 is one part in 10^6 (this does not include errors in input data). To compute the Mach number, it is necessary to find an expression for the speed of sound α. The expression that applies here is

$$\frac{1}{\alpha^2} = \left(\frac{\partial\rho}{\partial p}\right)_S = \left(\frac{\partial\rho}{\partial p}\right)_T + \left(\frac{\partial\rho}{\partial T}\right)_p\left(\frac{\partial T}{\partial p}\right)_S \quad (8)$$

The quantities $\left(\frac{\partial\rho}{\partial p}\right)_T$ and $\left(\frac{\partial\rho}{\partial T}\right)_p$ can be evaluated from the state equation, equation (1). The quantity $\left(\frac{\partial T}{\partial p}\right)_S$ is evaluated from equation (6). The resultant expression for $\frac{1}{\alpha^2}$ is

$$\frac{1}{\alpha^2} = \left(\frac{\partial\rho}{\partial p}\right)_S = \frac{1}{Z^2RT}\left[Z - p\left(\frac{\partial Z}{\partial p}\right)_T - \frac{\left[Z + T\left(\frac{\partial Z}{\partial T}\right)_p\right]^2}{\frac{c_{p0}}{R} - T\left(\frac{\partial}{\partial T}\left\{\int_0^p\left[Z - 1 + T\left(\frac{\partial Z}{\partial T}\right)_p\right]\frac{dp}{p}\right\}\right)_p}\right] \quad (9)$$

The actual Mach number is then given by

$$\mathsf{M}_1 = \frac{V_1}{\alpha_1} \quad (10)$$

The temperature correction, ΔT, to be added to the nozzle-throat temperature can be estimated by

$$\Delta T = -\frac{dT}{dM}\Delta M \quad (11)$$

where dT/dM is estimated by the perfect gas relations, and ΔM is the difference between the actual and desired Mach number. By using the new nozzle-throat temperature, a new Mach number is calculated. This process is repeated until the difference between the actual and desired Mach number is less than 10^{-5}. The mass-flow rate per unit area is then given by

$$(\rho V) = \frac{p_1}{Z_1RT_1}\cdot V_1 \quad (12)$$

and the ideal-gas mass-flow rate is

$$(\rho V)_{\text{ideal}} = \left\{\frac{2\gamma_i}{\gamma_i - 1}\cdot\frac{p_0^2}{Z_0RT_0}\left(\frac{p_1}{p_0}\right)^{\frac{2}{\gamma_i}}\left[1 - \left(\frac{p_1}{p_0}\right)^{\frac{\gamma_i-1}{\gamma_i}}\right]\right\}^{1/2} \quad (13)$$

Equation (13) is somewhat arbitrary. The presence of Z_0 insures that at a pressure ratio approaching unity, where the flow becomes incompressible, the mass-flow-rate defect (the difference between equations (12) and (13)), becomes zero. For both nitrogen and oxygen, γ_i was chosen to be 7/5. It is important to note that the choice of γ_i is arbitrary. Another choice would give different but equally valid results. The choice of 7/5 was based on the ideal model of a diatomic gas molecule with 5 degree of freedom.

The mass-flow rate for the critical-flow condition is that case in which the nozzle-throat Mach number is unity. The pressure ratio for the condition is the critical pressure ratio. The value for the ideal mass-flow rate for the critical-flow condition is

$$(\rho V)_{\text{critical, ideal}} = p_0\sqrt{\frac{\gamma_i}{Z_0RT_0}}\left(\frac{2}{\gamma_i + 1}\right)^{\frac{\gamma_i+1}{2(\gamma_i-1)}} \quad (14)$$

Case II: $Z = Z(\rho, T)$

This case applies to the remaining gases and to the Beattie-Bridgeman calculations. In all cases, the specific-heat data from reference [7] were used. The state equation for air and hydrogen is

$$Z = Z(\rho, T) = 1 + B_\rho\rho + C_\rho\rho^2 + D_\rho\rho^3 \quad (15)$$

For air, the virial coefficients B_ρ, C_ρ, and D_ρ are tabulated as functions of temperature in reference [7]. Analytic expressions for the virial coefficients were found in the same manner as those in Case I. For hydrogen, the compressibility factor Z is tabulated in reference [8] as a function of temperature and density. For a constant temperature, cubic equations in density were fitted to the compressibility data, which recovered the virial coefficients B_ρ, C_ρ, and D_ρ. Then, in a manner similar to that used for air, analytic expressions for the temperature variations of the virial coefficients were found.

The form of the Beattie-Bridgeman state equation used is

$$Z(\rho, T) = 1 + \left(B - \frac{A}{RT} - \frac{C}{T^3}\right)\rho + \left(-bB + \frac{aA}{RT} - \frac{CB}{T^3}\right)\rho^2 + \frac{CbB}{T^3}\rho^3 \quad (16)$$

The constants for the Beattie-Bridgeman state equation are from reference [10].

For steam, the state equation is that given in reference [9]. This equation readily reduces to the form $Z = Z(\rho, T)$.

The given quantities for these calculations are the inlet stagnation pressure p_0, the inlet stagnation temperature T_0, and the nozzle-throat Mach number $\mathbf{M}_1$. From the plenum conditions, the inlet stagnation density ρ_0 is calculated through the state equation. By assuming the gas to be perfect, an initial estimate of the nozzle-throat temperature T_1 can be made. The Mach number for this estimate is computed in the following manner.

constant R.) Equations (21) and (22) were solved for ρ_1 and for the internal-energy change of the gas. The computational accuracy of ρ_1 was one part in 10^6. The velocity of the gas in the nozzle throat V_1 is given by

$$V_1 = \left\{2R\left[\frac{U_0 - U_1}{R} + Z_0T_0 - Z_1T_1\right]\right\}^{1/2} \quad (23)$$

To compute the Mach number, it is necessary to derive an expression for the speed of sound α. The expression that applies for this case is

$$\alpha^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = \left(\frac{\partial p}{\partial \rho}\right)_T + \left(\frac{\partial p}{\partial T}\right)_\rho\left(\frac{\partial T}{\partial \rho}\right)_S \quad (24)$$

The quantities $\left(\frac{\partial p}{\partial \rho}\right)_T$ and $\left(\frac{\partial p}{\partial T}\right)_\rho$ can be evaluated from the state equation, $Z = Z(\rho, T)$. The quantity $\left(\frac{\partial T}{\partial \rho}\right)_S$ is evaluated from equation (21). The resulting expression for α^2 is

$$\alpha^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = RT\left[Z + \rho\left(\frac{\partial Z}{\partial \rho}\right)_T + \frac{\left[Z + T\left(\frac{\partial Z}{\partial T}\right)_\rho\right]^2}{\frac{c_{p0}}{R} - 1 - T\left(\frac{\partial}{\partial T}\left\{\int_0^\rho\left[Z - 1 + T\left(\frac{\partial Z}{\partial T}\right)_\rho\right]\frac{d\rho}{\rho}\right\}\right)_\rho}\right] \quad (25)$$

The differential entropy and internal energy (reference [11]) are:

$$dS = c_v\frac{dT}{T} - \frac{1}{\rho^2}\left(\frac{\partial p}{\partial T}\right)_\rho d\rho \quad (17)$$

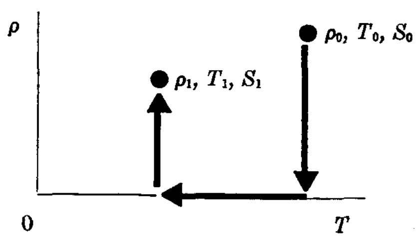
$$dU = T\,dS + \frac{p}{\rho^2}d\rho \quad (18)$$

If the pressure is eliminated through the state equation, the result is

$$\frac{dS}{R} = \frac{c_v}{R}\frac{dT}{T} - \left[Z + T\left(\frac{\partial Z}{\partial T}\right)_\rho\right]\frac{d\rho}{\rho} \quad (19)$$

$$\frac{dU}{R} = \frac{c_v}{R}dT - T^2\left(\frac{\partial Z}{\partial T}\right)_\rho\frac{d\rho}{\rho} \quad (20)$$

Equations (19) and (20) were integrated along the path indicated in the following sketch:



The equations in integral form become

$$\frac{S_1 - S_0}{R} \equiv 0 = \ln\frac{\rho_0}{\rho_1} - \int_{\rho_0}^{0}\left[Z - 1 + T\left(\frac{\partial Z}{\partial T}\right)_\rho\right]_{T=T_0}\frac{d\rho}{\rho} + \int_{T_0}^{T_1}\left(\frac{c_{p0}}{R} - 1\right)\frac{dT}{T} - \int_0^{\rho_1}\left[Z - 1 + T\left(\frac{\partial Z}{\partial T}\right)_\rho\right]_{T=T_1}\frac{d\rho}{\rho} \quad (21)$$

$$\frac{U_1 - U_0}{R} = -\int_{\rho_0}^{0}\left[T^2\left(\frac{\partial Z}{\partial T}\right)_\rho\right]_{T=T_0}\frac{d\rho}{\rho} + \int_{T_0}^{T_1}\left(\frac{c_{p0}}{R} - 1\right)dT - \int_0^{\rho_1}\left[T^2\left(\frac{\partial Z}{\partial T}\right)_\rho\right]_{T=T_1}\frac{d\rho}{\rho} \quad (22)$$

(The ideal-gas specific heat at constant volume equals the ideal-gas specific heat at constant pressure reduced by the specific gas constant R.)

The actual Mach number is given in equation (10). An iteration procedure similar to that used in Case I is used until the difference between the actual and desired Mach number is less than 10^{-5}.

The mass-flow rate is given by

$$(\rho V) = \rho_1V_1 \quad (26)$$

The nozzle throat pressure p_1 is calculated through the state equation. The ideal-gas mass-flow rate is still given by equation (13). The calculations for the critical-flow conditions are the same as those given in Case I. For the ideal-gas computations, the value of γ_i was chosen to be 7/5 for air and hydrogen, 5/3 for helium and argon, and 4/3 for steam. As in Case I, the values of γ_i are arbitrary. Another value would give different but equally valid results. The choices of 5/3, 7/5, and 4/3 are based on the ideal model of a monatomic, diatomic, and nonlinear triatomic gas molecule with 3, 5, and 6 degree of freedom, respectively.

Results and Discussion

Figs. 1 to 7 show the mass-flow-rate defect as a function of the pressure head divided by the stagnation pressure $(p_0 - p_1)/p_0$. The temperature for Figs. 1 to 6 is 550 deg R. The temperature for steam, Fig. 7, is 1500 deg R. The difference between the defects calculated from the virial equation and the Beattie-Bridgeman state equation is indicated where applicable. For the cases of helium and argon, only the Beattie-Bridgeman computations were made. For the case of steam, only the state equation of reference [9] was used.

It was found that when the difference between defects of the two computations was less than 0.001 the compressibility factor Z yielded by the virial equation was within 0.001 of the compressibility factor yielded by the Beattie-Bridgeman equation. Since the Beattie-Bridgeman calculation is more convenient, a possible criterion of when it may be used might be that the compressibility factor Z yielded by the virial equation be the same as that yielded by the Beattie-Bridgeman computations to the desired percentage accuracy. This presupposes that, in the ideal-gas calculation, the more accurate value of Z_0 is used.

Lines of constant Mach number are also shown in Figs. 1 to 7. With the exception of steam, Fig. 7, the defect is on the order of 0.002 at a Mach number of 0.2; the defect rises to a value between 0.01 and 0.04 for Mach 1 at a pressure of 100 atmospheres and temperature of 550 deg R. For steam, Fig. 7, the defect is about

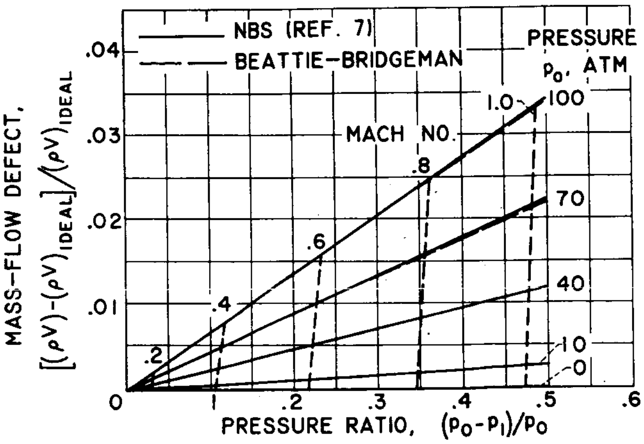


Fig. 1 Variation of mass-flow defect with pressure ratio for air at T_0 = 550 deg R using $\gamma_i = 7/5$

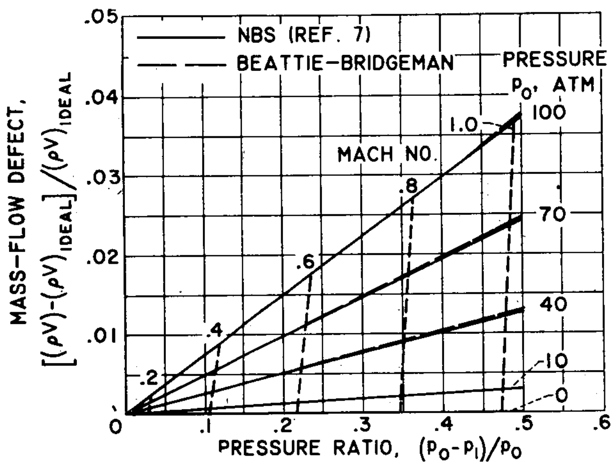


Fig. 2 Variation of mass-flow defect with pressure ratio for nitrogen at T_0 = 550 deg R and $\gamma_i = 7/5$

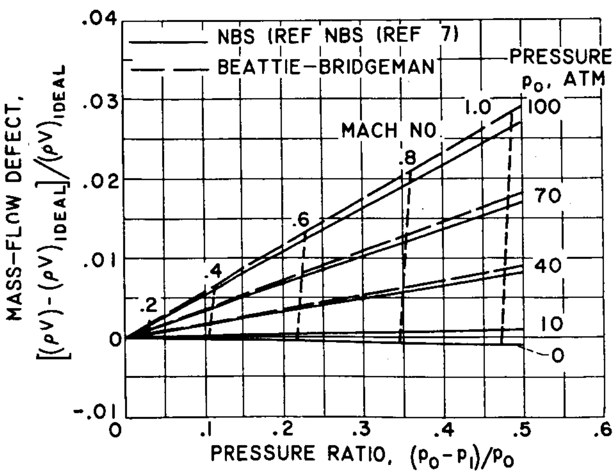


Fig. 3 Variation of mass-flow defect with pressure ratio for oxygen at T_0 = 550 deg R and $\gamma_i = 7/5$

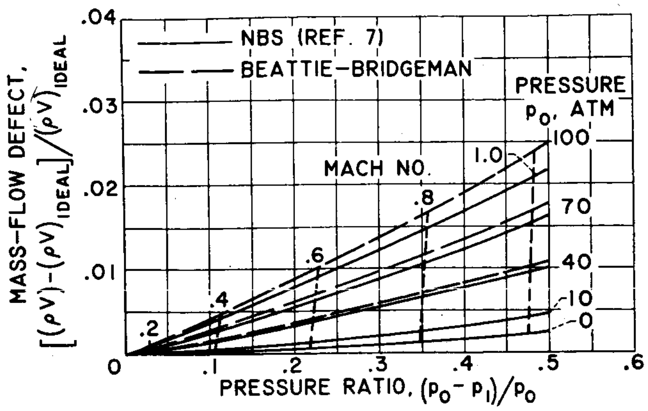


Fig. 4 Variation of mass-flow defect with pressure ratio for normal hydrogen at T_0 = 550 deg R and $\gamma_i = 7/5$

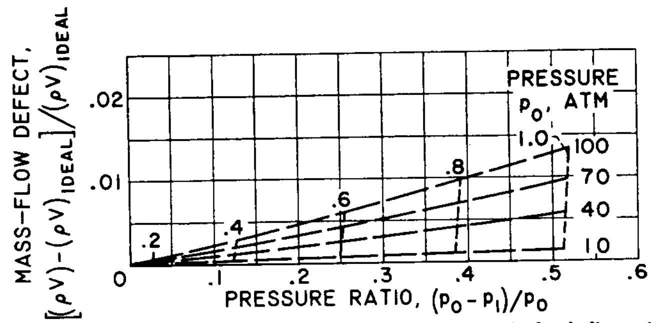


Fig. 5 Variation of mass-flow defect with pressure ratio for helium at T_0 = 550 deg R and $\gamma_i = 5/3$

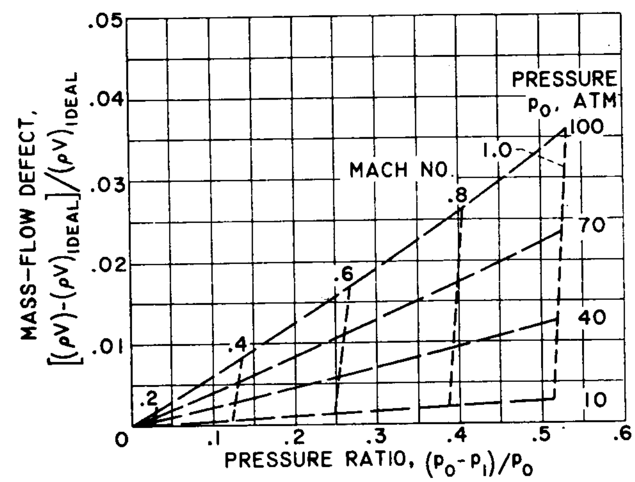


Fig. 6 Variation of mass-flow defect with pressure ratio for argon at T_0 = 550 deg R and $\gamma_i = 5/3$

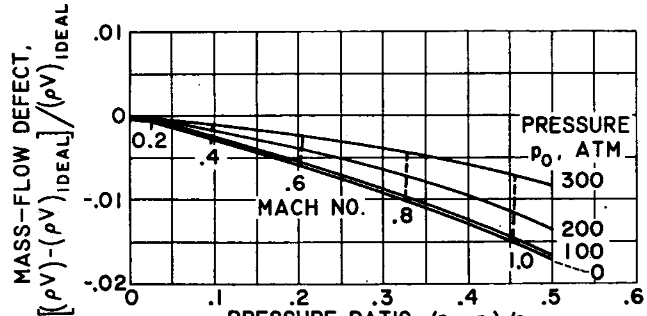


Fig. 7 Variation of mass-flow defect with pressure ratio for steam at T_0 = 1500 deg R and $\gamma_i = 4/3$

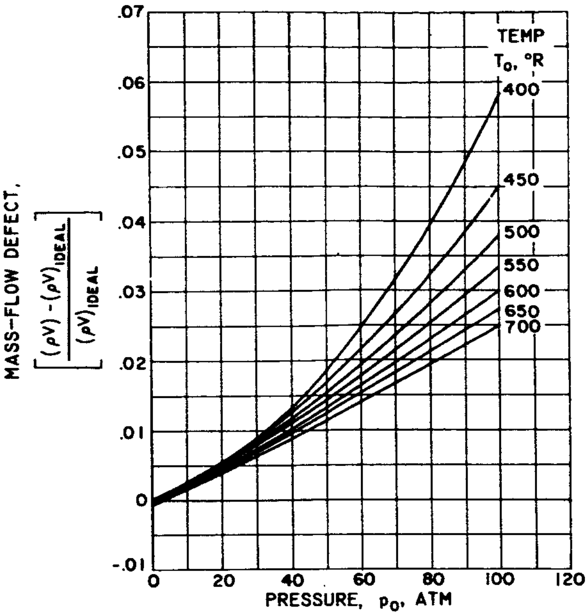


Fig. 8 Variation of mass-flow defect with pressure for critical flow of air and $\gamma_i = 7/5$

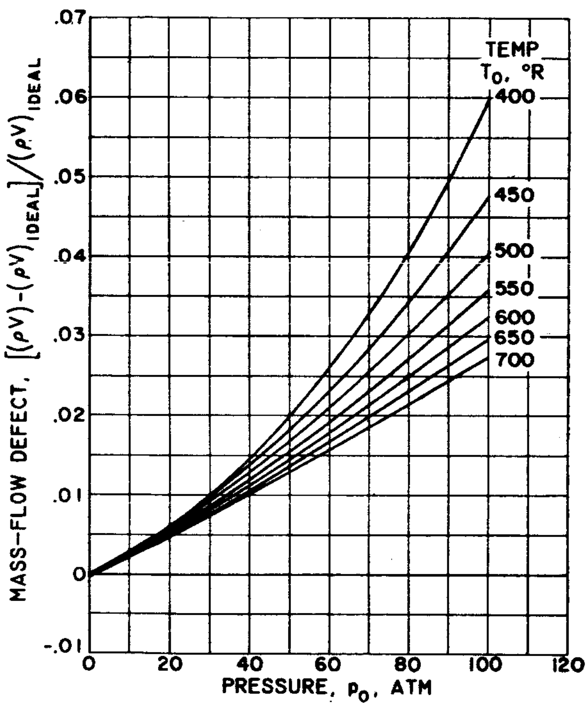


Fig. 9 Variations of mass-flow defect with pressure for critical flow of nitrogen and $\gamma_i = 7/5$

-0.001 for a Mach number of 0.2 and about -0.015 for a Mach number of unity.

Figs. 8 to 14 show the mass-flow-rate defect for critical flow in nozzles as a function of inlet stagnation pressure p_0. The parameter for these graphs is the inlet stagnation temperature T_0. For this case, the ideal mass-flow rate is calculated on the basis of the ideal critical-pressure ratio. It is observed that a defect of about 1/4 percent for air occurs at a pressure of 10 atm. This defect is of the same order as the accuracy of the discharge coefficient. Thus, even at 10 atm, the real-gas effects can be important.

The maximum defects range from about 0.02 for helium to about 0.06 for nitrogen. These defects occur at the high-pressure, low-temperature points.

For the cases of hydrogen and steam, Figs. 11 and 14, respectively, significant defects exist even at very low pressures.

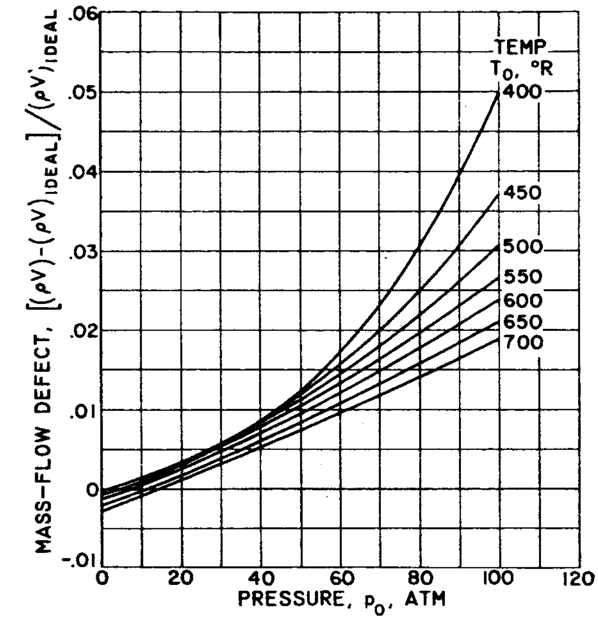


Fig. 10 Variation of mass-flow defect with pressure for critical flow of oxygen and $\gamma_i = 7/5$

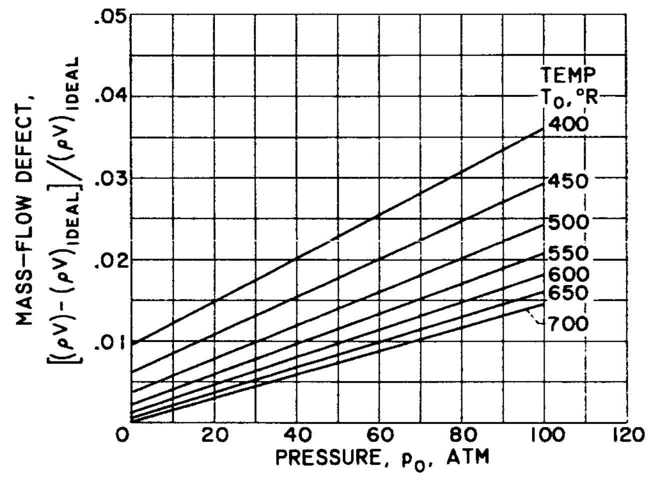


Fig. 11 Variation of mass-flow defect with pressure for critical flow of normal hydrogen and $\gamma_i = 7/5$

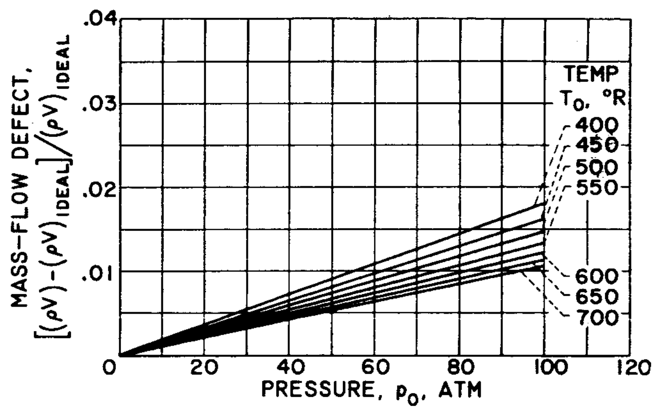


Fig. 12 Variation of mass-flow defect with pressure for critical flow of helium and $\gamma_i = 5/3$

These defects are not caused by variation in Z but are caused by the variation of the ideal-gas specific heats. At low pressures, the corrections would have been of a smaller magnitude if the actual value of γ_i at stagnation conditions had been used rather than the values $\gamma_i = 7/5$ for hydrogen and $\gamma_i = 4/3$ for steam. For

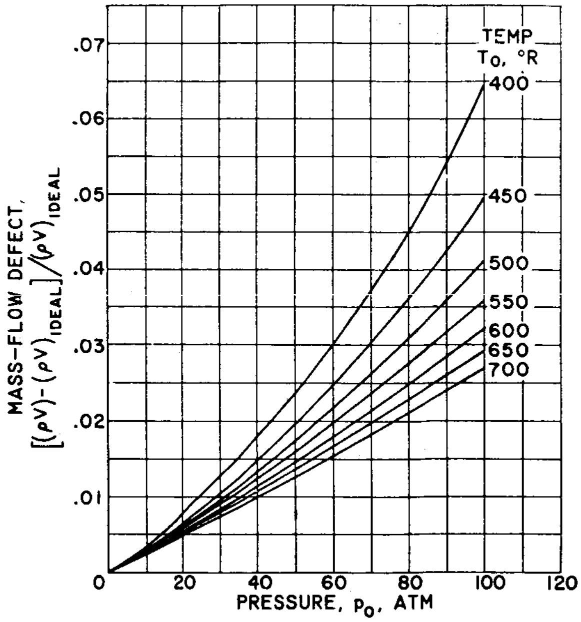


Fig. 13 Variation of mass-flow defect with pressure for critical flow of argon and $\gamma_i = 5/3$

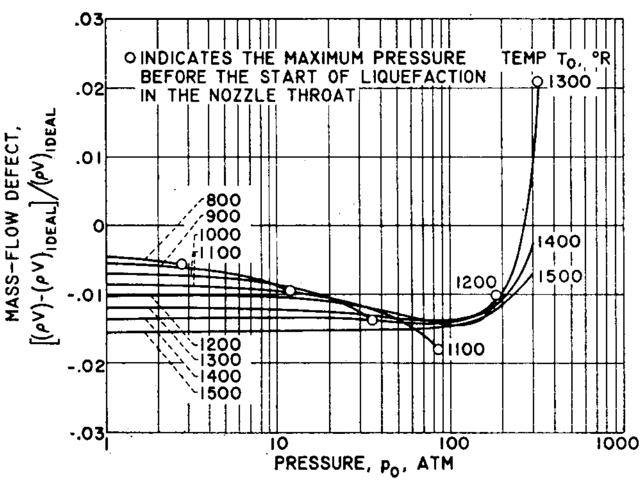


Fig. 14 Variation of mass-flow defect with pressure for critical flow of steam and $\gamma_i = 4/3$

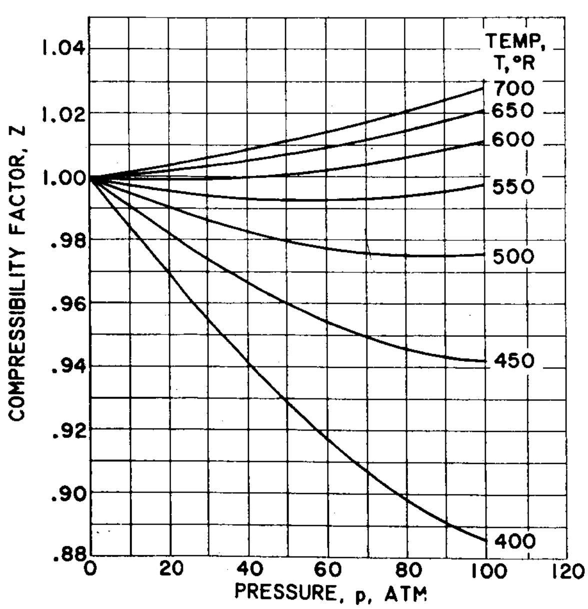


Fig. 15 Compressibility factor for air

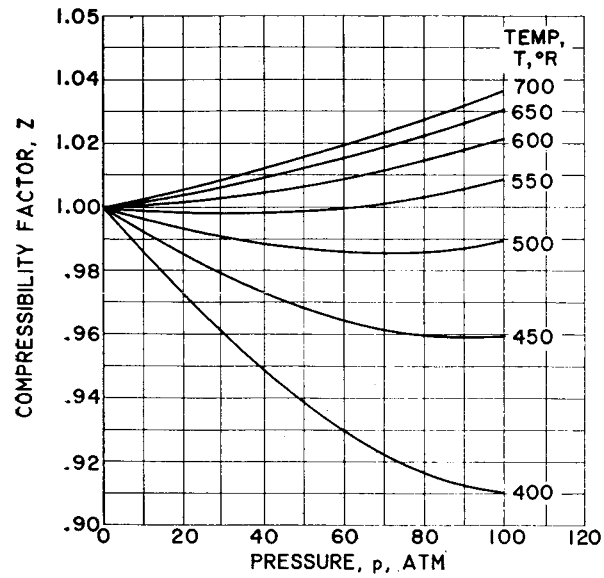


Fig. 16 Compressibility factor for nitrogen

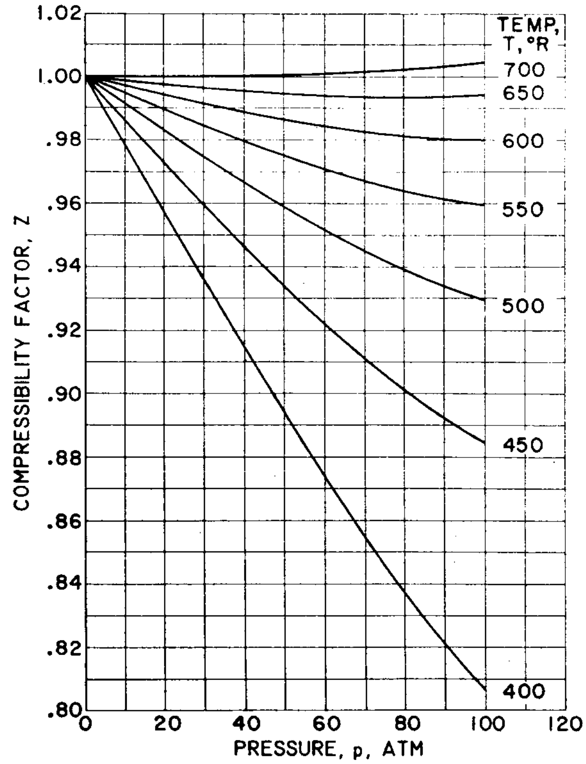


Fig. 17 Compressibility factor for oxygen

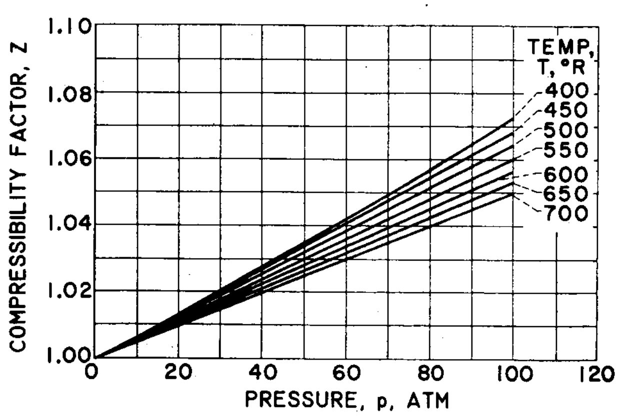


Fig. 18 Compressibility factor for hydrogen

DISCUSSION

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This paper is very welcome to the field of high pressure ratio gas flow measurement. Professor Arnberg's suggestion to include the compressibility factor Z-curves makes the paper a useful working tool because one can calculate the value of the "ideal" flow rate used in Figs. 1 to 14 in preparation for calculation of the actual flow rate without requiring other references. I prefer the use of the "perfect low pressure gas" flow rate for these figures to eliminate this intermediate step. Although the author points out the arbitrary definition of ideal gas flow in equation (13), and explains the presence of Z_0 (which may not be unity), it is inconsistent with his opening statement "a perfect gas will be defined as one whose specific heat is constant and independent of temperature and pressure and whose compressibility factor Z is unity . . . Ideal gas flow relations are defined as those that apply to a perfect gas." I mention this mainly to caution those readers who will not pay much attention to the calculation procedure.

Comparison of Fig. 8 with Fig. 4 in reference [6] shows that the inclusion of Z_0, in the ideal-gas critical flow-rate equation (13), accounts for about half the mass-flow defect between the constant gamma perfect gas with unity compressibility factor and the real gas. Hence if someone blindly uses Figs. 1 to 14 without careful attention to equation (13), a significant error results.

Equation (9) can be written in other forms which I present for informative purposes. Rewriting equation (9) as

$$\alpha^2 = \frac{ZRT}{1 - \dfrac{p}{Z}\left(\dfrac{\partial Z}{\partial p}\right)_T - \dfrac{\left[1 + \dfrac{T}{Z}\left(\dfrac{\partial Z}{\partial T}\right)_p\right]^2}{\dfrac{c_{p0}}{R} - T\left(\dfrac{\partial}{\partial T}\left\{\displaystyle\int_0^p \left[Z - 1 + T\left(\dfrac{\partial Z}{\partial T}\right)_p\right]\dfrac{dp}{p}\right\}\right)_p}} = kZRT \tag{28}$$

the denominator is recognized as the reciprocal of the isentropic exponent k from the definition of the speed of sound. Using an equation from reference [5]

$$\frac{\gamma}{k} = 1 - \frac{p}{Z}\left(\frac{\partial Z}{\partial p}\right)_T \tag{29}$$

where γ is the ratio of specific heats and k is the isentropic exponent, and the easily derived equation

$$\gamma = \frac{1 - \dfrac{p}{Z}\left(\dfrac{\partial Z}{\partial P}\right)_T}{1 - \dfrac{p}{Z}\left(\dfrac{\partial Z}{\partial P}\right)_T - \dfrac{ZR}{c_p} - \dfrac{2\mu p}{T} - \dfrac{c_p}{ZR}\left(\dfrac{\mu p}{T}\right)^2} \tag{30}$$

where μ is the Joule Thomson coefficient

$$\mu = \frac{RT^2}{pc_p}\left(\frac{\partial Z}{\partial T}\right)_p \tag{31}$$

we can write the denominator of equation (28) as

$$k = \frac{1}{1 - \dfrac{p}{Z}\left(\dfrac{\partial Z}{\partial p}\right)_T - \dfrac{ZR}{c_p} - \dfrac{2\mu p}{T} - \dfrac{c_p}{ZR}\left(\dfrac{\mu p}{T}\right)^2} \tag{32}$$

Hence the last three denominator terms of equation (32) should be equal to the last term of equation (9). $\dfrac{ZR}{c_p}$ is the most significant term.

Because of the significance of $(\partial Z/\partial p)_T$ and k, they should be tabulated in publications along with other gas properties.

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Author's Closure

Mr. Reimer is quite correct when he warns the reader about the use of Figs. 1 through 14 without paying careful attention to equation (13). Again, I wish to iterate, the calculations of the mass-flow rate by means of equations (13) or (14) not only require knowledge of the mass-flow defect as given in Figs. 1 through 14, but also knowledge of the plenum compressibility factor as given in Figs. 15 through 21.

I feel that a comment concerning accuracy is in order. The computational uncertainty in calculating the mass-flow rate by the methods in this paper is about 0.1 percent which is on the order of the uncertainty of the physical factors involved in this analysis.

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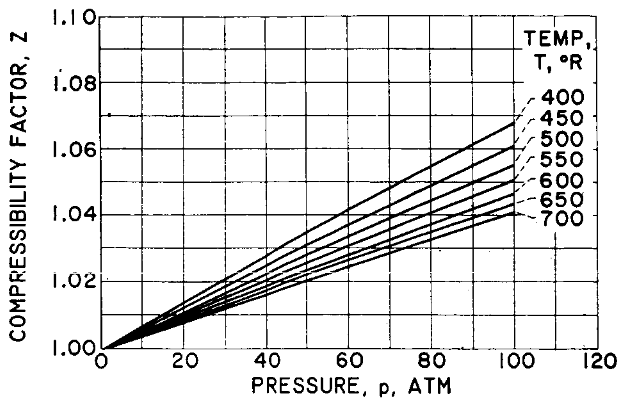

Fig. 19 Compressibility factor for helium

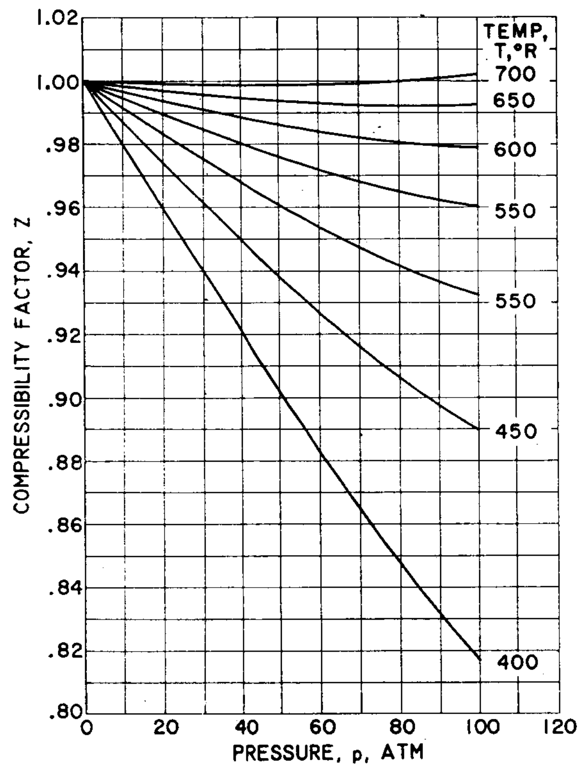

Fig. 20 Compressibility factor for argon

the case of steam, the endpoint of the curve is indicated by a small circle in those cases where a higher pressure would initiate liquefaction at the nozzle throat.

By using Figs. 8 to 14, an estimate can be made of the mass-flow-rate defect for pressure ratios other than critical. This estimate, whose accuracy is on the order of $^1/_4$ percent, is

$$\text{mass-flow defect} = 2.0\,\frac{(p_0 - p_1)}{p_0} \times (\text{mass-flow defect for critical pressure ratio}) \quad (27)$$

Values of compressibility factor Z as functions of pressure and temperature for the various gases are plotted in Figs. 15 through 21.[2] These values are given as an aid in computing the ideal-gas mass-flow rate represented by equations (13) and (14).

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[2] These graphs are included at the suggestion of B. T. Arnberg. Values of Z for all gases but helium are from reference [7]; values for helium are computed from the Beattie-Bridgeman constants given in reference [10].

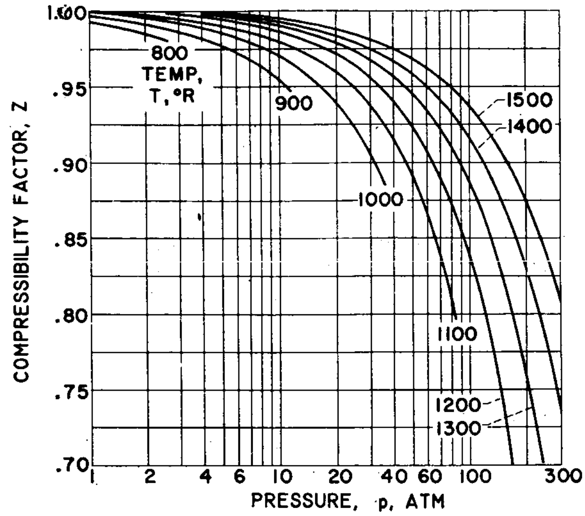

Fig. 21 Compressibility factor for steam

Summary

Computer calculations of real-gas effects in critical-flow nozzles have been made for air, nitrogen, oxygen, hydrogen, argon helium, and steam. The results indicate that for critical flow of air, at room temperature and 100 atm pressure, real-gas effects of $3^1/_2$ percent exist. Similar magnitudes are found for the other gases.

It was also found that the agreement between the mass-flow rate defect calculated by the use of the Beattie-Bridgeman state equation and that calculated by use of a more exact equation was on the same order as the agreement between the compressibility factor calculated by use of the Beattie-Bridgeman state equation and that calculated by use of the more exact state equation.

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